Rigidity percolation in a field

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Rigidity percolation with g degrees of freedom per site is analyzed on randomly diluted Erdös-Renyi graphs, with average connectivity γ, in the presence of a field h. In the (γ,h) plane, the rigid and flexible phases are separated by a line of first-order transitions whose location is determined exactly. This line ends at a critical point with classical critical exponents. Analytic expressions are given for the densities n_F of uncanceled degrees of freedom and γ_r of redundant bonds. Upon crossing the coexistence line, γ_r and n_F are continuous, although their first derivatives are discontinuous. We extend, for the case of nonzero field, a recently proposed hypothesis, namely, that the density of uncanceled degrees of freedom is a ''free energy'' for rigidity percolation. Analytic expressions are obtained for the energy, entropy, and specific heat. Some analogies with a liquid-vapor transition are discussed. Particularizing to zero field, we find that the existence of a $(g+1)$ core is a necessary condition for rigidity percolation with g degrees of freedom. At the transition point γ_c, Maxwell counting of degrees of freedom is exact on the rigid cluster and on the $(g+1)$ rigid core, i.e., the average coordination of these subgraphs is exactly $2g$, although γ_c, the average coordination of the whole system, is smaller than $2g$. γ_c is found to converge to $2g$ for large g, i.e., in this limit Maxwell counting is exact globally as well.

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I. INTRODUCTION

Scalar percolation (SP) [1,2] is a paradigm for the geometric phase transition that takes place on an initially disconnected lattice of pointlike sites when the density p of present bonds is continuously increased. At the percolation point p_c the system becomes connected on a macroscopic scale. This may mean that transport can happen across the system (conductivity, fluid flow), or that the system becomes correlated on macroscopic scales, i.e., ordered. It is in fact possible to describe magnetic transitions in terms of the percolation of properly defined clusters [3–5]. Because of the generality and simplicity of the concepts involved, this paradigm has found multiple applications in science [1,6,2,7]. In most cases, the physical variables attached to sites are scalars, i.e., each site has one associated *degree of freedom*.

A generalization of this paradigm considers the case in which there is more than one degree of freedom per site, and has been termed rigidity percolation (RP) [8–15]. In g-RP, each site of a lattice has g degrees of freedom, and each present bond eliminates one relative degree of freedom. The most commonly invoked application of RP deals with the statics of structures. Consider, for example, the problem of bracing a *framework* in three dimensions, i.e., rigidly connecting a set of pointlike joints by means of rotatable bars. Each joint has three translational degrees of freedom, $g=3$, and each rotatable bar fixes the distance between two nodes, thus providing one relative constraint. The question of whether a given set of bars is enough to rigidize a given structure constitutes a classical problem in applied mathematics, that of graph rigidity [16–18]. In statistical physics, bonds (bars) are randomly present with probability p or absent with probability $1-p$, and one asks for the typical rigid properties of such structures. Upon increasing the density p of present bonds, the system goes into the rigid phase, characterized by the existence of an extensive rigidly connected cluster. The RP problem becomes fully equivalent to SP when $g=1$. SP has a continuous transition in all dimensions [1]. In two dimensions the RP transition is continuous but in a different universality class [19] from SP. In the mean field (MF) limit, RP ($g>1$) has a first-order transition [20,21].

The situation is reminiscent of the Potts model, whose MF transition is continuous for $q=2$ and discontinuous for $q>2$ [22,23]. Potts models in the presence of a field, and their relation with percolation models, have been studied recently because of possible links with the deconfining transition in QCD [24–26]. In the presence of a field, and for large enough q, the Potts model has a line of first-order transitions ending at a critical point [27,28]. This critical point appears to be always in the Ising universality class.

Nonzero field values have been considered in scalar percolation studies previously [29–35]. A field may be introduced in percolation by allowing for the existence of ''ghost bonds'' which are present with probability h and connect sites directly to a solid background (or to ''infinity''). However, for any nonzero field there is no SP transition. RP is somewhat more interesting, as we will find out. In this work, RP is studied in diluted random graphs of the Erdös-Renyi-type, with average connectivity γ, in the presence of a field h. Unlike SP, mean-field RP has, in the presence of a field, a line of first-order phase transitions. This line ends at a critical point with classical critical indices $\alpha=0$, $\beta=1/2$, $\tilde{\gamma}=1$, and $\delta=3$.

Some of the analysis in zero field is relevant for the related problem of bootstrap percolation (BP) [36–39], also known as k core [40] in the field of graph theory [41–43]. In k BP [36], all sites with less than k neighbors are iteratively culled. What remains, if something, is the k core [40]; a

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subgraph where all sites have k or more neighbors. In g-RP a site needs at least g bonds in order to be attached to a rigid cluster. Thus the ''infinite'' rigid cluster is a subset of the g core. In Sec. II B we will see that an even stronger condition exists for rigidity: g-RP requires the existence of a $(g+1)$ core [44].

Our approach starts by deriving an equation of state for the ''order parameter'' $R(\gamma,h)$, the probability that a randomly chosen site belongs to the rigid cluster, as a function of γ, the average number of bonds impinging on a site, and $h=\gamma H$, where H is the applied field. We will call H or h indistinctly the ''field'' variable. The equation of state, as is customary in these cases, is found to accept multiple solutions. Stability analysis is not enough to single out a unique solution. Of central importance, in order to lift this multiplicity, are $\gamma_r(\gamma,h)$, the average number of ''redundant'' bonds per site (see later), and n_F, the average number of uncanceled degrees of freedom per site. Their relevance resides in the fact that they *must* be continuous functions of γ. Requiring that n_F (or, equivalently, γ_r) be continuous is enough to identify the physically correct solution.

This work is similar in spirit to previous treatments of RP in zero field on Bethe lattices [21], i.e., networks where each site has exactly z randomly chosen neighbors. Here we consider the effect of an external field and particularize to random graphs of the Erdös-Renyi-type [45]. The introduction of a field appears to be much more tractable analytically on Erdős-Renyi graphs than on Bethe lattices, and this is the main reason why most of the results presented here are for Erdős-Renyi graphs. For these, we are able to derive an analytic expression for the densities n_F of uncanceled degrees of freedom and γ_r of redundant bonds for arbitrary values of the field. Bethe lattices are discussed briefly in Appendix D. In zero field, analytic calculations are also straightforward for Bethe lattices, and allow us to obtain, in an entirely analytic fashion, some of the results obtained by numerical integration in Ref. [21]. Bethe lattices with a field are also discussed, however the calculations become rapidly cumbersome in this case.

It is known that SP and Potts models are particular cases of a more general Fortuin-Kasteleyn random-cluster model, defined by a continuous parameter q [46,47]. SP can be obtained as the $q\to 1$ limit of this random-cluster model and, in this limit, the logarithm of the partition function coincides with the average number of connected clusters. It seems possible that a similar mapping might exists for RP as well, although it has not been found up to now. However it has been proposed [48,21] that the number of uncanceled degrees of freedom n_F is a good ''free-energy'' candidate for RP in zero field. When $g=1$, each connected cluster has one uncanceled degree of freedom, so both definitions coincide in this limit. We explore on this idea further in this work.

It is possible to establish a pedagogical parallel between the RP transition and a condensation transition. One identifies the coordination parameter γ with an inverse temperature β; the negative of the order parameter R plays the role of the fluid volume V (it is also possible to identify R with ρ, the fluid density) [49] and the field $H=h/\gamma$ is the fluid pressure P. Within this analogy, it is natural to argue that requiring the continuity of n_F in order to identify the physically correct solution R in rigidity percolation is equivalent to requiring the continuity of the free energy (giving rise to the Maxwell construction) in the statistical mechanics treatment of condensation transitions at the MF level [50].

It is shown in this work that the idea of identifying n_F with a free energy for RP leads to consistent results in the presence of a field as well. In Sec. VI it is shown that (1) the order parameter R is obtained as a derivative of the free energy with respect to the field, (2) the condition of stability (Sec. III C) may be related to the positivity of a suitable second derivative of the free energy, and (3) the continuity of n_F can be cast exactly in the form of Maxwell's rule of equal areas on the RP equivalent of a P-V diagram (which is the H-R diagram, see Fig. 7).

This work is organized as follows. The system under consideration is defined in Sec. II A, and its equation of state is derived in Sec. II B, establishing its connections with bootstrap percolation. In Sec. II C the field is introduced. Section III starts the analysis of solutions of the general equation of state, discussing stability and the existence of a critical point. The concept of redundant constraints is introduced in Sec. IV A and their density is calculated in Sec. IV B. This result is used in Sec. V to determine the value γ_c where the first-order transition takes place. In Sec. V B, the counting of constraints is done on the $(g+1)$-rigid core in zero field. Section VI discusses several consequences of identifying the density of uncanceled degrees of freedom with a free energy, and Sec. VII contains a discussion of the results. Bethe lattices are briefly considered in Appendix D.

II. SETUP

A. Randomly diluted graphs

We consider graphs made of N sites (or ''nodes'') where each of the $N(N-1)/2$ pairs of distinct sites is connected by a bond (or ''edge'') independently with probability p. This defines [45] an Erdös-Renyi graph with average coordination number $\gamma=p(N-1)$. In this work γ will be taken to be of the order of 1. For large N, a site of this graph is connected to k other sites with Poissonian probability $P_k(\gamma)=e^{-\gamma}\gamma^k/k!$.

As appropriate for rigidity percolation, each node of this graph is regarded as a ''body'' with g degrees of freedom. For example, rigid bodies in d dimensions have d translational degrees of freedom plus $d(d-1)/2$ rotational degrees of freedom for a total of $g=d(d+1)/2$. In this work, g is taken to be an arbitrary integer. Each present bond represents one *constraint*, that is, removes one degree of freedom. In order to have a physical representation in mind, a bond can be thought of as a rotatable bar that fixes the distance between two arbitrary points belonging to the bodies it connects [51,15]. For the class of graphs that we consider, at most one bond is allowed for each pair of nodes.

As the number of present bonds grows, parts of this graph will become ''rigidly connected.'' Rigid connectivity of a subgraph means (this is our definition) that the total number of degrees of freedom in the system cannot be further reduced by additional bonds connected between nodes of this

subgraph. Such subgraphs are customarily called ''rigid clusters.'' A rigid cluster has no internal degrees of freedom left. When the coordination γ is large enough, the largest rigid cluster encloses a finite fraction of the system's sites, and rigidity is said to percolate. For low γ the system is in the flexible, or floppy, phase, and there are no extensive rigid clusters.

A rough estimate of the threshold for the appearance of an extensive rigid cluster can be obtained by equating the average number of constraints per site, which is $\gamma/2$, to g. This results in the so-called Maxwell estimate [13,52,48,21], $\gamma_c^{Maxwell}=2g$. This estimate becomes exact in the $g\gg 1$ limit (Sec. V B).

B. The equation of state in zero field

Let us now define $\mathcal{C}$ to be the largest rigid cluster in the graph, if one exists. A site in $\mathcal{C}$ will be said to be a ''rigid site.'' Let R be the probability that a randomly chosen node i be in $\mathcal{C}$. In order for i to be in $\mathcal{C}$ it is necessary and sufficient that it be connected to g or more other nodes j in $\mathcal{C}$. Our definition of rigidity is thus *recursive* at this stage.

A site j is rigid and connected to i with probability pR. Therefore i has exactly k rigid neighbors with probability $\binom{N-1}{k}(pR)^k(1-pR)^{N-1-k}$. For N large and defining $x=\gamma R$, the probability P_k to have exactly k rigid neighbors may be written as

$$P_k(x)=e^{-x}x^k/k!, \tag{1}$$

showing that the number k of *rigid neighbors* of a randomly chosen site is a Poissonian variable with average $x=\gamma R=\sum_{k=0}^{\infty}kP_k(x)$. Since a site must have g or more rigid neighbors in order to be rigid itself, we conclude that R satisfies the self-consistent equation

$$R=\sum_{k=g}^{\infty}P_k(\gamma R)=G_g(\gamma R), \tag{2}$$

where we have defined

$$G_m(x)=\sum_{k=m}^{\infty}P_k(x). \tag{3}$$

Bootstrap percolation, or k core

Let us now briefly discuss the related problem of bootstrap percolation [36] or k core [40]. We want to assess the probability $P_{\mathrm{BP}}(g+1)$ that a randomly chosen site i be part of the $(g+1)$ core. Assume this is the case. By following one of its links, a neighbor j is reached. j must have at least g other neighbors. We call these neighbors of j, other than the site i from which we arrived at it, the ''outgoing'' neighbors of j. Each of these in turn must have g or more outgoing neighbors, and so on. More formally, let us define the property of g-outgoing connectedness (g-OC) in the following (recursive) way: A site is g-OC if g or more of its outgoing neighbors also are.

On graphs of the Erdös-Renyi-type, the probability to have k outgoing neighbors is the same as that to have k neighbors altogether, i.e., $P_k(\gamma)$ [Eq. (1)]. This is so because links are independently present. Letting $\hat{R}$ be the probability that a site be g-OC, by the same reasoning as in the preceding section we conclude that a random site is connected to exactly k outgoing g-OC neighbors with probability $P_k(\gamma\hat{R})$. Thus $\hat{R}$ satisfies the same self-consistent equation (2) as R does in the RP problem with g degrees of freedom, i.e.,

$$\hat{R}=G_g(\gamma\hat{R}). \tag{4}$$

Note that by requiring that each site in a tree has g or more g-OC neighbors we ensure that all sites, except perhaps for the top one, have $(g+1)$ or more neighbors. The probability $P_{\mathrm{BP}}(g+1)$ that the top site itself has $g+1$ or more neighbors (which are g-OC) is then given by

$$P_{\mathrm{BP}}(g+1)=G_{g+1}(\gamma\hat{R})=\hat{R}-P_g(\gamma\hat{R}). \tag{5}$$

This expression gives the density of the $(g+1)$-BP infinite cluster, [or $(g+1)$ core] at the point where it first appears [36,41,43]. So one must first solve Eq. (4) in order to obtain $\hat{R}$ as a function of γ, and then use Eq. (5) to find $P_{\mathrm{BP}}(g+1)$. Numerical results [53] show that Eq. (5) is exact for large N.

We see that the $(g+1)$-core density $P_{\mathrm{BP}}(g+1)$ is somewhat smaller than $\hat{R}$ whenever $\hat{R}>0$, while $\hat{R}$ in turn satisfies the same equation as the density R of rigid sites in g-RP. Later in Sec. V B we will see that in zero field, whenever there is a g-rigid cluster, it contains as a subset the $(g+1)$ core.

C. Equation of state in the presence of a ghost field

Building on ideas first discussed by Essam [34], we now introduce a ''ghost field'' H that couples to the order parameter R in the following way. In addition to the ''normal'' bonds of our graph, we will have ghost bonds. Each ghost bond connects one randomly chosen site to a ''rigid background'' [54], and provides *one* constraint, i.e., removes one degree of freedom. The total number of ghost bonds in the system is by definition $N\gamma H=Nh$. Multiple occupation is allowed, so that the probability for a site to have n ghost bonds is Poissonian: $P_n(h)=e^{-h}h^n/n!$. (See Appendix A for a discussion of the differences between Poissonian and linear field definitions.)

If $n\leqslant g$ ghost bonds are connected to a site, then this site has $g-n$ degrees of freedom left. Otherwise if $n\geqslant g$, this site has no degrees of freedom left, i.e., it is rigidly attached to the background. It is easy to see that all *extensive* rigid clusters are rigidly connected to the background with probability one, if $h>0$. From now on, a site is said to be rigid if it is rigidly connected to the background, either directly through ghost bonds, or indirectly through rigid neighbors.

If a site has $n\geqslant g$ ghost edges connecting it to the background, then it is rigid. Otherwise if $n<g$, it is rigid if it has, in addition to these n ghost bonds, $(g-n)$ or more rigid

neighbors. Thus using Eq. (1) we may write, in the presence of a field $h=H\gamma$,

$$R=\sum_{n=g}^{\infty} P_n(h)+\sum_{n=0}^{g-1} P_n(h)\sum_{k=g-n}^{\infty} P_k(\gamma R). \tag{6}$$

After a simple resummation, this expression reads

$$R=G_g(\gamma R+h), \tag{7}$$

with G_g given by Eq. (3). Equation (7) generalizes Eq. (2) in the presence of a field, and is the equation of state for our problem.

The simplicity of Eq. (7) is one of the reasons leading us to study this particular (Poissonian) definition of the field. Other field definitions (see Appendix A), for example, assuming that each site is rigidly attached to the background with probability h, or other random graph structures such as Bethe lattices [21] (see Appendix D) are also tractable with the methods used here, but the algebra becomes more complicated.

Clearly $y=\gamma(H+R)$ plays the role of a ''Weiss field'' in the MF equation for a ferromagnet. By analogy we may thus identify γ as the inverse temperature and H as the magnetic field. It is illustrative to take notice of the similarities between Eq. (7) and other MF equations. For the Potts model [55],

$$m=\Theta_q(\beta m+h), \tag{8}$$

where β is the inverse temperature, m is the magnetization, $h=\beta H$ is the external field, and $\Theta_q(y)=(e^y-1)/[e^y+(q-1)]$.

Note that when $g=1$ (scalar percolation), Eq. (7) gives $G_1(y)=1-e^{-y}$, the same as Eq. (8) for $q=1$. This is of course just a consequence of the known equivalence between scalar percolation and the $q\to 1$ limit of the Potts model [46,47].

However, the archetypal example of a first-order transition with a two-parameter phase space is the condensation transition [50], described at the mean-field level by the van der Waals equation. The reduced form of the van der Waals equation can be written as

$$\lambda=\Gamma(\beta(\lambda+P)), \tag{9}$$

where $\Gamma(y)=27y^2/(y+8)^2$, P is the pressure, β is the inverse temperature, and $\lambda=3\rho^2$ with ρ the number density. Γ has the same general features as G and Θ, namely, it starts at zero, grows sharply, and saturates for large values of its argument. Therefore all three systems give rise to the same phenomenology, including, of course, sharing the same (classical) critical indices at their critical points.

Within an analogy with a condensation transition, in the RP problem γ plays the role of an inverse temperature, while $H=h/\gamma$ plays the role of a pressure.

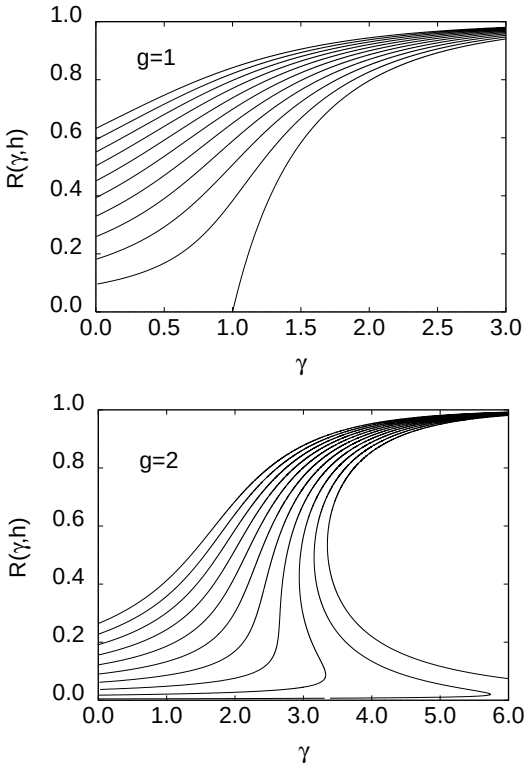

FIG. 1. The density R of rigid sites as given by Eq. (7) for $g=1$ (scalar percolation, top) and $g=2$ (rigidity percolation, bottom). The field takes the values $h=0.0,0.1,0.20,\ldots,1.0$. For $h=0$, $R=0$ is a solution for all γ in both cases. For $g=2$ the critical field is $h_c=0.287$.

III. ANALYSIS OF SOLUTIONS

In this section, the solutions of our equation of state (7) are discussed. In order to obtain $R(\gamma,h)$ numerically for given γ and h, one might, for example, iterate Eq. (7) until a desired numerical accuracy is reached. This procedure was used in Refs [20,21]. However in this work the following alternative procedure was preferred: given $h>0$ fixed, and for a sequence of values of $y>h$, we evaluate

$$R=G_g(y)=G_g(x+h),$$

$$\gamma=(y-h)/G_g(y)=x/G_g(x+h), \tag{10}$$

thus defining $R(\gamma,h)$ implicitly. In this way one obtains the results displayed in Fig. 1. This procedure allows us to obtain all solutions of Eq. (7), while the iterative procedure mentioned above only provides the stable branches.

In zero field, $R=0$ is a solution $\forall\ \gamma$ and $\forall\ g$. If γ is large enough, nontrivial solutions $R>0$ exist as well. If $h>0$, one has on the other hand that there is at least one nontrivial solution $R>0\ \forall\ \gamma$.

In the SP case ($g=1$), $G_1(x)=1-\exp(-x)$. If h is nonnegative, Eq. (10) always has two solutions $R(\gamma,h)$ for all $\gamma>0$. If $h=0$ there is, in addition to the trivial solution $R=0$, a nontrivial solution that is negative when $\gamma<\gamma_c=1$ and becomes positive for $\gamma>\gamma_c=1$. Above γ_c one has $R\approx(\gamma-1)$. In this case there is a *continuous* transition at γ

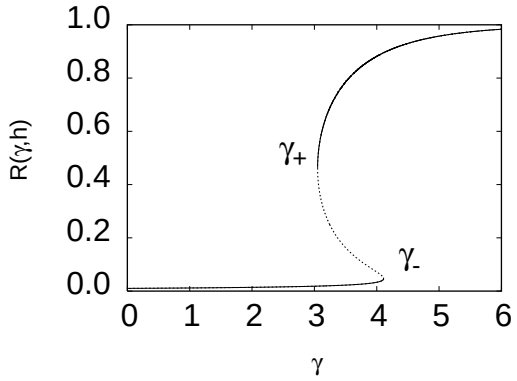

FIG. 2. For all $g>1$, the order parameter R as given by Eq. (10) becomes multivalued for small values of the field h. The branch joining the spinodals $\gamma_{\pm}$ (dashed) has $dR/d\gamma<0$ and is thus unstable. The uppermost and lowermost branches (solid) are stable. In this example, $g=2$ and $h=0.15$. For zero field the spinodal γ_{-} goes to infinity and the branch $0\leqslant\gamma\leqslant\gamma_{-}$ collapses onto the $R=0$ solution, which is stable for all γ.

$=1$, a well known result for scalar percolation on random graphs [45]. If $h>0$, there are also two branches for $R(\gamma,h)$, however only one of them is positive. The other branch is negative and therefore unphysical. Thus there is no transition (Fig. 1) for $h>0$.

A richer behavior is found for the RP case ($g>1$), as shown in Fig. 1 for $g=2$. In this case γ, as given by Eq. (10), may no longer be a monotonous function of y, and, for this reason, R becomes a multivalued function of γ. In contrast to SP, where R is multivalued but only one solution is positive, here both are. This allows for the existence of a first-order transition. The physical considerations which lead to the identification of the correct transition point will be discussed in Secs. III C and V.

A. Spinodal points

The condition that γ as given by Eq. (10) be stationary in $x=\gamma R$ reads

$$G_g(x+h)-x\frac{\partial G_g(x+h)}{\partial x}=0, \tag{11}$$

and has two solutions $x^s_{\pm}(g,h)$ for all h smaller than a critical value $h^*(g)$. These two solutions in turn define $\gamma^s_{\pm}$, which are turning points for $R(\gamma,h)$ (see Fig. 2). In the interval $\gamma^s_{+}<\gamma<\gamma^s_{-}$, $R(\gamma,h)$ has three solutions, two of which are stable as we show next. Therefore a *discontinuous* γ-driven transition takes place when $h<h^*(g)$. Right at $h=h^*$ the transition becomes continuous (see Sec. III B), while for $h>h^*(g)$, R is a smooth function of γ and no transition occurs.

The thresholds $\gamma^s_{\pm}$ can be identified as *spinodal points*, as first discussed in Ref. [21]. The true rigidity percolation transition happens at a value $\gamma_c(g,h)$ that lies in between the spinodals, and which we analytically determine later in Sec. V.

B. Critical point

1. Scalar percolation ($g=1$)

For $g=1$ the critical field is $h^*=0$. An additional condition for criticality is that x/γ be tangential to $G(x)$, and has a unique solution $\gamma^*=1$ and $R^*=0$. In the scalar case there is a continuous transition at $\gamma_c=1$ [45].

2. Rigidity percolation ($g>1$)

When $g>1$, the condition that x/γ be tangent to $G_g(y)$ [Eq. (11)] identifies the spinodal points, and is not enough to single out the critical point. However, when h takes its critical value h^*, the spinodals $x^s_{\pm}$ coalesce onto an inflexion point. The *critical point* $\{h^*,\gamma^*,R^*\}$ is thus defined by

$$\left.\frac{\partial^2 G_g}{\partial x^2}\right|_{x^*,h^*}=0, \tag{12a}$$

$$\left.\frac{\partial G_g}{\partial x}\right|_{x^*,h^*}=\frac{G_g(x^*+h^*)}{x^*}=\frac{1}{\gamma^*}. \tag{12b}$$

For $h>h^*$ there is no phase transition. Using Eq. (7), and defining $y=x+h$, these two equations can be solved exactly. The critical point turns out to be

$$y^*=g-1,$$

$$\gamma^*=\{P_{g-1}(g-1)\}^{-1}=e^{g-1}(g-1)!/(g-1)^{g-1},$$

$$R^*=G_g(g-1)=e^{1-g}\sum_{k=g}^{\infty}(g-1)^k/k!,$$

$$h^*=y^*-x^*=g-1-\gamma^*R^*. \tag{13}$$

For $g=2$ [56] one finds

$$\gamma^*=e,$$

$$R^*=(e-2)/e,$$

$$h^*=3-e. \tag{14}$$

For large g, and approximating $n!\approx(n/e)^n(2\pi n)^{1/2}$, one sees that $\gamma^*\propto g^{1/2}$, while $h^*\sim g$ and thus $H^*\sim g^{1/2}$. This means that in the limit $g\gg 1$ most constraints are field constraints at the critical point.

3. Critical indices

The RP transition on random graphs is similar to other MF transitions with first-order lines, as discussed in Sec. II C. Thus it can be concluded that RP must have classical critical indices $\beta=1/2$, $\delta=3$, and $\tilde{\gamma}=1$. For completeness we show that this is indeed the case by deriving the critical indices briefly in Appendix C.

C. Stability analysis

Stability can be analyzed if Eq. (7) is interpreted as a recursion relation,

$$R_{t+1}=G(\gamma R_t+h), \quad t=0,1,\ldots,\infty, \tag{15}$$

for fixed γ and h. Assume $R(\gamma,h)$ is a fixed point of Eq. (15), i.e., $R=G(\gamma R,h)$. This fixed point is stable if

$$\left|\frac{\partial G(\gamma R+h)}{\partial R}\right|_{\gamma,h}\Bigg| = \left|\gamma\frac{\partial G(x+h)}{\partial x}\right|_{x=\gamma R}\Bigg| < 1. \tag{16}$$

Since $G'(y)=P_{g-1}(y)$ we conclude that the stability condition reads

$$\gamma P_{g-1}(y)<1, \tag{17}$$

with $y=x+h$ as defined previously.

A more useful form results if one notes that, if $R>0$, Eq. (16) is equivalent to requiring that $dR/d\gamma>0$. In fact,

$$\frac{dR}{d\gamma}=\frac{dG}{dx}\frac{dx}{d\gamma}=G'\frac{d}{d\gamma}(\gamma R)=G'\left(R+\gamma\frac{dR}{d\gamma}\right)$$

$$\Rightarrow \quad \frac{dR}{d\gamma}(1-\gamma G')=G'R \tag{18}$$

and since $G'>0\ \forall\ x>0$, we conclude that a nontrivial fixed point $R(\gamma,h)>0$ of Eq. (15) is stable if and only if $dR/d\gamma>0$.

This condition has a simple physical meaning: increasing the average connectivity γ should not decrease the rigid density R. A similar stability condition holds for fluids, namely, that the coefficient of thermal expansion be positive.

1. Scalar percolation ($g=1$)

In zero field, $R=0$ is a solution of Eq. (7), thus a fixed point of Eq. (15), $\forall\ g$. If $g=1$, stability [Eq. (17)] requires that $\gamma P_0=\gamma e^{-\gamma R}<1$. Thus the trivial solution $R=0$ becomes unstable for $\gamma>1$, where the nontrivial solution (stable because $dR/d\gamma>0$, see Fig. 1) first appears. This situation is typical of continuous transitions; the ordered solution appears exactly at the point where the paramagnetic solution becomes unstable.

When $h>0$ there is only one solution for Eq. (7), and it is stable for all γ since $dR/d\gamma>0$ [Eq. (18)].

2. Rigidity percolation ($g>1$)

For $g>1$ the situation is more interesting, as several stable solutions of Eq. (7) can coexist. If $h<h^*$, Eq. (7) has three solutions (see Fig. 2) in the range $\gamma_-^s>\gamma>\gamma_+^s$. The branch joining the two spinodal points can be discarded since it is unstable (because $dR/d\gamma<0$). However the other two branches (solid lines in Fig. 2) are stable so that there is coexistence of two stable solutions for $\gamma_-^s>\gamma>\gamma_+^s$. Thus the system undergoes a γ-driven *first-order* transition somewhere between the spinodals. The precise point γ_c at which the system switches from one stable solution to the other is uniquely defined by a continuity requirement, as we elaborate later in Sec. V.

The $h=0$ case is similar, however in this case the lowest stable branch collapses onto the trivial solution $R=0$, while the spinodal γ_- goes to ∞.

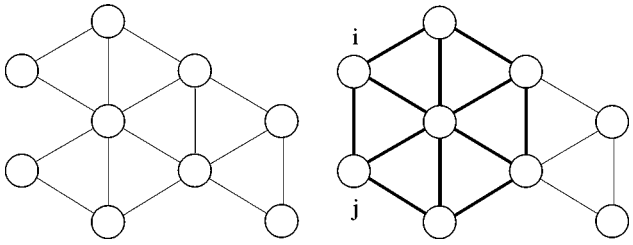

FIG. 3. In this two-dimensional example, each node (circle) represents a point that has two positional degrees of freedom, while each bond fixes the distance between two points and thus provides one constraint. The graph on the left still has three remaining degrees of freedom: two translations and one rotation. Bond ij (right) is a redundant bond, because it connects two nodes which were already rigidly connected. After adding bond ij, the bonds that provided the rigid connection between i and j become *overconstrained* (thick lines). Any one of these can be removed without altering the number of remaining degrees of freedom, which still equals three. The graph on the right has one redundancy, but 12 overconstrained bonds.

IV. ZERO MODES, AND REDUNDANT AND OVERCONSTRAINED BONDS

In this section the counting of uncanceled degrees of freedom and the useful notions of redundant and overconstrained bonds are discussed. These will be of central importance in our subsequent treatment of the RP problem.

A. Definitions

We consider a graph made of N sites, each with g degrees of freedom, and for the moment assume that $h=0$, i.e., there are no ghost-field constraints. Each present bond removes one degree of freedom from the system, unless it is a *redundant* bond. A redundant bond (or constraint) is the one that links two nodes which were *already* rigidly connected, for example, nodes i and j in Fig. 3. The addition of a redundant constraint does not reduce the number of degrees of freedom in the system. Thus the balance of degrees of freedom reads

$$N_F=Ng-E+B_r, \tag{19}$$

where N_F is the number of uncanceled degrees of freedom [57], E is the number of bonds in the graph, and B_r is the number of redundant bonds.

For an Erdös-Renyi graph one has $\langle E\rangle=\gamma N/2$. Defining $\langle B_r\rangle=\gamma_r N/2$, the average density $n_F(\gamma)$ of zero modes per site is written as

$$n_F(\gamma)=\frac{\langle N_F\rangle}{N}=g-\frac{1}{2}[\gamma-\gamma_r(\gamma)]. \tag{20}$$

Let us assume that b is a redundant bond. This means that its two end nodes are rigidly connected even if b is removed. Let $\mathcal{B}_b$ be the subset of bonds that, in the absence of b, provide rigidity to its two end nodes. After adding b, any of the bonds in $\{\mathcal{B}_b+b\}$ (thick lines in Fig. 3) can be removed without altering the total number of degrees of freedom N_F. The subset of bonds $\{\mathcal{B}_b+b\}$ is said to be overconstrained. From a physical point of view, the overconstrained bonds may be defined as those that carry an internal stress because of the addition of a redundant bond b that has a length mismatch.

Please notice the important difference between the number of redundancies and the number of overconstrained bonds: when adding a redundant bond (such as ij in Fig. 3) the number of redundancies B_r always increases by exactly one. However, the number of overconstrained bonds may increase by more than one. In the example shown in Fig. 3, the number of overconstrained bonds increases by 12.

We then conclude that, when adding *any* bond to a graph, the number of redundancies B_r will either increase by one (if the chosen sites were rigidly connected) or stay unchanged (if not), in which case the number of zero modes N_F will either stay unchanged or decrease by one. This implies that the densities γ_r of redundant bonds and n_F of zero modes must be continuous functions of the density γ of present bonds. The density of overconstrained bonds, on the other hand, need not be continuous.

Let us now discuss how the balance of degrees of freedom is modified by ghost-field bonds (Sec. II C). If a site has $n<g$ present ghost edges connecting it to the rigid background, then it contributes with $(g-n)$ degrees of freedom to Eq. (19). If on the other hand it is connected to the background by $n\geqslant g$ ghost bonds, then this site has no degrees of freedom left to contribute to Eq. (19). The effective number $g_{\mathrm{eff}}(h)$ of remaining degrees of freedom contributed by an average site is thus

$$g_{\mathrm{eff}}(h)=\sum_{n=0}^{g}(g-n)P_n(h)=g[1-G_{g+1}(h)] -h[1-G_g(h)], \tag{21}$$

where G_g is defined by Eq. (3). Taking this result into account, in the presence of a field h, we now have that

$$n_F(\gamma,h)=g_{\mathrm{eff}}(h)-\tfrac{1}{2}[\gamma-\gamma_r(\gamma,h)]. \tag{22}$$

Note that n_F and γ_r are trivially related through the addition of a known function of γ and h. Therefore, the knowledge of either n_F or γ_r provides exactly the same physical information about the system.

The importance of Eq. (22) is twofold. In the first place, as discussed above, n_F and γ_r must be continuous functions of γ. A similar reasoning allows one to conclude that they must also be continuous functions of h. The continuity of γ_r and n_F is a key property, and will be used later in Sec. V to select the physically correct solution of Eq. (7) whenever indeterminacies arise. Second, in Sec. VI it will be argued that n_F can be identified with the logarithm of the partition function for the RP problem, and the consequences of such identification will be discussed. Although this identification is not necessary for solving the RP problem, it provides interesting additional insight into this problem, insofar it helps making a link with thermodynamics.

B. Calculation of γ_r

We now show how the density $\gamma_r(\gamma,h)$ of redundant bonds is calculated. Obviously when $\gamma=0$ there are no redundant bonds, i.e., $\gamma_r(\gamma=0,h)=0$, so we can write

$$\gamma_r(\gamma,h)=\int_0^{\gamma}\left.\frac{\partial\gamma_r}{\partial\gamma}\right|_h d\gamma, \tag{23}$$

where the integral is done along a path of constant h. In Appendix B we show [58] that

$$\gamma\left.\frac{\partial\langle B_r\rangle}{\partial\gamma}\right|_h=\langle B_{ov}\rangle, \tag{24}$$

where $\langle B_{ov}\rangle_\gamma$ is the total number of overconstrained bonds in $\mathcal{B}$. Consider two randomly chosen sites i and j. As discussed in Sec. IV, a bond b_{ij} is overconstrained if i and j are rigidly connected to the background, even in the absence of this bond. On a random graph, this happens with probability R^2. Thus a bond is present and overconstrained with probability $(\gamma/N)R^2$, and therefore the average number of overconstrained bonds is $\langle B_{ov}\rangle=\gamma R^2N/2$. We can now write Eq. (24) as

$$\left.\frac{\partial\gamma_r}{\partial\gamma}\right|_h=R^2, \tag{25}$$

so that now Eq. (23) reads

$$\gamma_r(\gamma,h)=\int_0^{\gamma}R^2(\gamma,h)d\gamma. \tag{26}$$

Note that, while Eqs. (23) and (24) are exact for arbitrary graphs, Eq. (25) only holds under the assumption that two neighboring sites i and j are independently rigid with probability R. Thus in deriving Eq. (26), correlations have been ignored. This is correct on random graphs of the type considered here, but fails on finite-dimensional lattices.

It is not immediately obvious how Eq. (26) can be integrated, since we do not have an explicit expression for R as a function of γ. However using $x=\gamma R=y-h$ one can write $R^2d\gamma=Rdx-xdR=[R-(y-h)\partial R/\partial y]dy$, where $R=G_g(y)$ as given by Eq. (7). After integrating by parts,

$$\gamma_r=\left\{2\int G_g(y)dy-(y-h)G_g(y)\right\}\Big|_{y=h}^{y=x+h}. \tag{27}$$

Using Eq. (3), Eq. (27) can be integrated to give

$$\begin{aligned}\gamma_r&=\{(y+h-2g)G_g(y)+2gP_g(y)\}\big|_{y=h}^{y=x+h}\\&=(y+h)G_g(y)-2hG_g(h)\\&\quad+2g[G_{g+1}(h)-G_{g+1}(y)].\end{aligned} \tag{28}$$

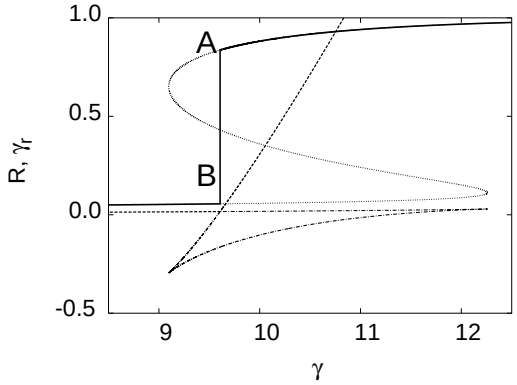

FIG. 4. The density of redundant bonds γ_r (thick dashed) describes a Maxwell loop (dot-dashed) whose crossing point defines the location of the true transition. After discarding the unphysical loop, the density R of rigid sites (thick solid) jumps at the transition between A and B, but γ_r is continuous. Shown is an example with $g=10$ and $h=5$.

This expression enables us to calculate the density of redundant bonds analytically. In previous work on Bethe lattices [21], Eq. (26) was solved in zero field by numerical integration, using for this purpose the values of R that result from iteration of Eq. (2). Although the algebra becomes slightly more complicated, the approach used here to calculate γ_r is applicable to Bethe lattices as well. Later in Sec. V B we present some results for Bethe lattices.

1. The density of zero modes

Using Eqs. (7), (21), (22), and (28) we can write

$$n_F(\gamma,h)=\frac{\gamma}{2}(R^2-1)+(h-g)(R-1)+gP_g(y), \quad (29)$$

where $y=x+h=\gamma R+h=\gamma(R+H)$. Because of Eq. (22), continuity of γ_r implies that of n_F.

2. A consistency check

In the limit $\gamma\to N$, all bonds are present and there are no remaining degrees of freedom, i.e., $n_F\to 0$. It is easy to verify that Eq. (29) indeed satisfies this condition.

V. LOCATION OF THE TRANSITION POINT

A. The general case

The value $\gamma_c(h)$ where the transition takes place is univocally determined by the requirement that γ_r be a continuous function of γ. In order to see how this works, imagine that for a given $h<h^*$ one lets $h<y<\infty$ and calculates $\gamma_r(y,h)$ [Eq. (28)] and $\gamma(y,h)$ [Eq. (10)], thus obtaining a plot of γ_r vs γ (Fig. 4). In doing so one finds that γ_r describes a loop (sometimes called a ''Maxwell loop'' in statistical mechanics).

The existence of a crossing point implies that two values y_A and y_B exist such that

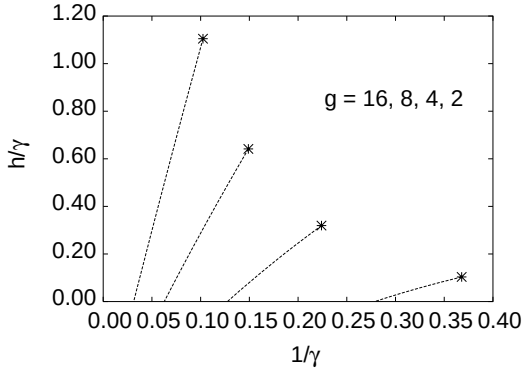

FIG. 5. Coexistence lines (lines of first-order phase transitions—dashed) in the space defined by the two intensive parameters of rigidity percolation, the field $H=h/\gamma$ and the ''temperature'' $1/\gamma$, for several values of the number g of degrees of freedom per site. The critical point, defined by Eq. (13), is indicated with an asterisk. For all $g\geqslant 2$, i.e., for rigidity percolation, this critical point has classical indices $\alpha=0$, $\beta=1/2$, $\tilde{\gamma}=1$, and $\delta=3$. The case $g=1$ (scalar percolation, not shown) has its critical point at $\gamma=1$ and $H=0$, and belongs to a *different* universality class, with $\alpha=-1$, $\beta=1$, $\tilde{\gamma}=1$, and $\delta=2$.

$$\{(y+h-2g)G_g(y)+2gP_g(y)\}\big|_{y=y_A}^{y=y_B}=0, \quad (30)$$

where $y_{A,B}=\gamma_c(h)R_{A,B}+h$.

Equation (30) cannot be solved analytically in general, however it is easily solved numerically (by iteration) for each value of h and g, to obtain $\gamma_c(g,h)$. In this fashion the location of the coexistence line can be calculated to arbitrary precision (Fig. 5). At the transition point $\gamma_c(h)$ the order parameter R jumps [Fig. 6(a)], but the density of redundant bonds γ_r is continuous [Fig. 6(b)].

B. The case of zero field: Maxwell counting on the $(g+1)$-rigid core

For zero field one has that $R=0$ for $\gamma<\gamma_c$. Equation (26) then implies that $\gamma_r=0$ for $\gamma\leqslant\gamma_c$. At the transition point γ_c, a rigid cluster suddenly appears that contains a fraction R_c of the graph. In zero field, Eq. (30) reads

$$(x_c-2g)R_c+2gP_g(x_c)=0, \quad (31)$$

where all quantities are evaluated in the rigid phase (R and y are zero in the floppy phase).

Even this simpler equation does not admit analytic treatment. However, very precise solutions can be obtained numerically as described already. Table I contains some examples.

Please note that γ_c is *smaller* than the value $2g$ needed for global balance of constraints and degrees of freedom. However, all degrees of freedom belonging to sites *on the rigid cluster* (which appears suddenly at the transition point) are by definition canceled out by constraints. Moreover, since for zero field the number of redundant constraints per site is exactly zero at the transition (see Fig. 6), one concludes that the average number of bonds per site, for sites on

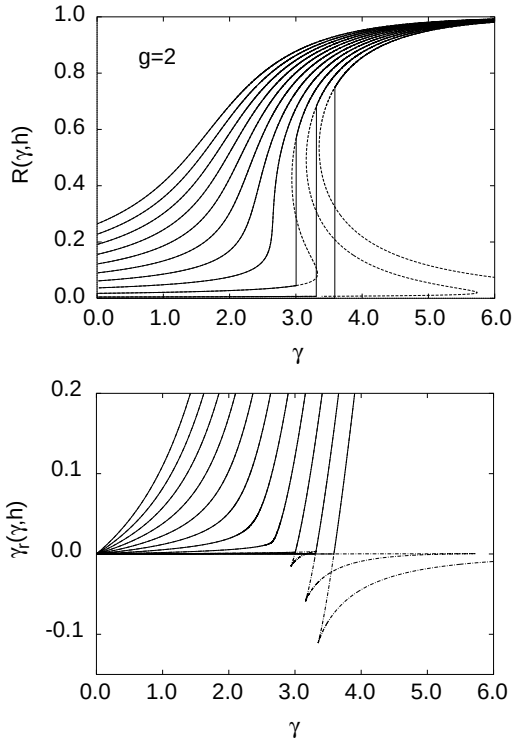

FIG. 6. (a) The final values for the density of rigid sites R (thick lines). The transition point $\gamma_c(h)$, where R undergoes a discontinuous jump, is determined by the physical requirement that the density of redundant bonds γ_r (b) be continuous. In this example $g=2$ and (from right to left) $h=0.0,0.1,0.20,\ldots,1.0$. For $g=2$ the critical field is $h_c=0.287$.

the rigid cluster, must be exactly $2g$ at γ_c. The conclusion is then that sites on the rigid cluster have more bonds than average.

This observation may be formalized in the following way. Using $\gamma R=x$, $R=\Sigma_{k=g}^{\infty}P_k(x)$ with $P_k(x)=e^{-x}x^k/k!$, and $xP_k(x)=(k+1)P_{k+1}(x)$, condition (31) can be written as

$$\sum_{k=g+1}^{\infty}(2g-k)P_k(x)=0, \tag{32}$$

which in turn implies

TABLE I. The rigidity percolation threshold coordination γ_c, and the jump in the rigid cluster density R_c on Erdös-Renyi random graphs with zero field, obtained from solving Eq. (31) for several values of g, the number of degrees of freedom per site.

g	$\gamma_c(g)$	Discontinuity R_c
2	3.58804747296539 . . .	0.74915378510250 . . .
3	5.7549256115462 . . .	0.8812398370507 . . .
4	7.84295819428849 . . .	0.933538491167168 . . .
5	9.89551361859910 . . .	0.959638549916489 . . .
6	11.9288724790262765 . . .	0.974275720961391 . . .
$\to\infty$	$2g$	1.0

$$\frac{\sum_{k=g+1}^{\infty}kP_k(x)}{\sum_{k=g+1}^{\infty}P_k(x)}=2g. \tag{33}$$

This sum counts the number of rigid neighbors for sites on the $(g+1)$-rigid core, that is, the subgraph of the rigid cluster that has minimum coordination $(g+1)$. Equation (33) means that the $(g+1)$-rigid core has an exact balance of degrees of freedom.

Clearly, the rigid cluster may also contain sites not in the $(g+1)$-rigid core. When these are considered, it is possible to see that the balance of constraints and degrees of freedom is still respected.

The case of large g

An analytic solution of Eq. (31) is possible (though somewhat trivial) in the $g\gg1$ limit. In this case $P_g\to0$ and $R_c\to1$, i.e., there is a jump from $R=0$ to $R=1$ at the transition point. Plugging these observations into Eq. (31) one gets, for zero field,

$$\gamma_c=2g, \qquad g\gg1. \tag{34}$$

This means that the rigid transition happens exactly at the point where global balance between constraints and degrees of freedom is attained. In other words, at γ_c Maxwell counting is exact on the rigid cluster for all g, while for $g\gg1$ it is globally exact as well.

VI. RELATION BETWEEN n_F AND A FREE ENERGY

In this section, the consequences of making the identification $n_F\to(\ln\mathcal{Z})/N$ are explored. It has been shown some time ago that SP can be mapped onto the $q\to1$ limit of the Potts model [46,47]. This mapping allows one to draw a parallel between an equilibrium thermodynamic transition (Potts) and percolation, which may be described as a purely geometric transition. One of the outcomes of this equivalence is the identification of the total number of clusters in percolation as the logarithm of the partition function for the resulting Potts model. The existence of this mapping has had profound impact on our understanding of scalar percolation, e.g., by allowing the exact calculation of percolation critical indices in two dimensions [2]. No equivalent mapping has been found for RP yet. However it has been suggested [48,21] that a possible generalization for the free energy in RP for zero field is the total number of remaining degrees of freedom (zero modes or *floppy modes*). In the $g=1$ case of RP each cluster has one remaining degree of freedom, thus the SP result is recovered.

In Sec. IV it was shown that n_F (equivalently γ_r) has to be continuous. This was used in Sec. V to locate the point γ_c where the first-order transition happens. Thus n_F plays a role similar to that of the Gibbs free energy, which is also a continuous function of its intensive parameters. In this section we add further evidence supporting the identification of n_F as the logarithm of the "partition function," or free energy,

for the RP problem on random graphs. We demonstrate that the *ansatz* $\mathcal{Z}_{RP} \propto e^{N n_F}$ is consistent also in the presence of a field, and show how several thermodynamic quantities result from derivatives of this free energy.

In Appendix it is shown that the derivatives of n_F are

$$\left.\frac{\partial n_F}{\partial \gamma}\right|_h = \frac{1}{2}(R^2-1), \tag{35a}$$

$$\left.\frac{\partial n_F}{\partial h}\right|_\gamma = R-1. \tag{35b}$$

As expected, the order parameter R results form deriving the logarithm of the partition function with respect to the field-like parameter, up to a constant shift.

From Eqs. (35) we furthermore obtain

$$\left.\frac{\partial n_F}{\partial \gamma}\right|_H = \frac{1}{2}(R^2-1)+H(R-1), \tag{36a}$$

$$\left.\frac{\partial n_F}{\partial H}\right|_\gamma = \gamma(R-1). \tag{36b}$$

A. Maxwell's rule of equal areas

The continuity requirement (30) employed in this work to locate the true transition point γ_c is analogous to the condition that the free energy be continuous, as noted in previous work [21]. The continuity of an appropriate thermodynamic potential is discussed in elementary statistical mechanics textbooks in connection, for example, with van der Waals' equation [50,59], where it is used as a criterion to locate the transition. The continuity condition for our RP problem is written in general as

$$\int_A^B dn_F = 0, \tag{37}$$

where A and B are the two phases that coexist at the first-order transition (see Fig. 4). Clearly Eq. (30) is the particular case of Eq. (37) for which the integration is done along a path of constant field. Choosing a path $A \to B$ on which γ is constant instead, it is easy to put Eq. (37) in the form of the famous Maxwell's "rule of equal areas." Starting from

$$0 = \int_A^B \left.\frac{\partial n_F}{\partial h}\right|_\gamma dh, \tag{38}$$

where the integral is done along an "isotherm" (a line of constant γ). After using Eq. (35b) and integrating by parts, this reads

$$h(R_B - R_A) = \int_A^B h(R,\gamma)\, dR, \tag{39}$$

where $h(R,\gamma) = G_g^{-1}(R) - \gamma R$.

Equation (39) is equivalent to the condition $P(V_B - V_A) = \int_A^B P(V,T)\, dV$ for a fluid, i.e., Maxwell's rule. Figure 7 serves to illustrate the fact that the areas above and below the horizontal coexistence line are equal, as Eq. (39) requires.

B. Energy, entropy, and work

In the following we will assume that n_F [Eq. (29)] is the logarithm of the partition function, i.e., $n_F = -\gamma f$, with f a free-energy density and $\gamma = 1/T$ the inverse temperature. The energy density e is then

$$e = -\left.\frac{\partial n_F}{\partial \gamma}\right|_H = \frac{1}{2}(1-R^2) + H(1-R). \tag{40}$$

The entropy per site s turns out to be

$$s = -\left.\frac{\partial f}{\partial T}\right|_H = n_F + \gamma e = g[P_g(y) - (R-1)]. \tag{41}$$

From the first law of thermodynamics, and using Eqs. (40) and (41), we conclude that, in an infinitesimal transformation, the "work" done by the system against the environment is:

$$dw = dq - de = \frac{1}{\gamma} ds - de = \frac{1}{\gamma}\frac{\partial n_F}{\partial H} dH = (R-1)\, dH. \tag{42}$$

The analog of the constant-pressure heat capacity for a fluid is in our case the constant-field heat capacity

$$c_H = -\gamma \left.\frac{\partial s}{\partial \gamma}\right|_H = -\gamma^2 \left.\frac{\partial e}{\partial \gamma}\right|_H = \frac{y^2 P_{g-1}(y)}{1-\gamma P_{g-1}(y)}, \tag{43}$$

which is non-negative by Eq. (17), and diverges as $|\gamma - \gamma^*|^{-1}$ at the critical point. This of course should not be taken to mean that $\alpha = 1$, since it is the constant-density heat capacity what defines α (For a discussion, see Ref. [60].) The constant-field heat capacity c_H is expected to diverge with the same exponent $\tilde{\gamma}$ as the susceptibility, unless the system has certain symmetries, which is the case for the Ising model, but not for RP.

The analogous of the "constant-density" heat capacity is

$$c_R = -\gamma \left.\frac{\partial s}{\partial \gamma}\right|_R, \tag{44}$$

which is zero and thus $\alpha = 0$.

C. Clausius-Clapeyron equation

Let $\Delta f = -1/\gamma \Delta n_F = -1/\gamma (n_F(B) - n_F(A))$ be the free-energy difference between the two phases in the coexistence region. This is a function of the temperature $T = 1/\gamma$ and the field $H = h/\gamma$. Following standard texts [61], we write

$$\left.\frac{\partial H}{\partial T}\right|_{\Delta f} = -\frac{\left.\frac{\partial \Delta f}{\partial T}\right|_H}{\left.\frac{\partial \Delta f}{\partial H}\right|_T} = -\frac{\Delta s}{\Delta R}, \tag{45}$$

where the derivative on the left-hand side is evaluated on any line of constant Δf. Using Eq. (41) one can write $\Delta s = g(\Delta P_g - \Delta R)$. On the coexistence line, $\Delta f = 0$ so $\Delta s|_{coex} = \gamma \Delta e|_{coex}$ and, using Eq. (40) we have that $\Delta s|_{coex} = -h\Delta R - \gamma/2\Delta(R^2) = -\Delta R[(y_A + y_B)/2]$. Thus the Clausius-Clapeyron equation for the RP problem can be written as

$$\left.\frac{\partial H}{\partial T}\right|_{coex} = \frac{y_A + y_B}{2}. \tag{46}$$

Since on approach to the critical point, $y \to y^* = g - 1$ [Eq. (13)], we conclude that the coexistence line reaches the critical point with slope $(g-1)$ in the H-T plane (see Fig. 5). Equivalently by a simple change of variables one can write Eq. (46) as

$$\left.\frac{\partial h}{\partial \gamma}\right|_{coex} = -\frac{R_A + R_B}{2}, \tag{47}$$

implying that the coexistence curve approaches the critical point with slope $-R^*$ in the h-γ plane.

VII. DISCUSSION

The rigidity percolation problem with g degrees of freedom per site has been considered on Erdös-Renyi graphs with average coordination γ. An external field h is introduced by connecting each site to a rigid background, or ''ghost site,'' with a certain probability that depends on h as described in Sec. II C. The resulting equation of state (7) for the density R of sites that are rigidly connected to the background undergoes a first-order phase transition on a ''coexistence line'' $\gamma_c(h)$. This line ends at a critical point (Fig. 5) with classical exponents $\alpha = 0$, $\beta = 1/2$, $\tilde{\gamma} = 1$, and $\delta = 3$. For comparison, the critical point of scalar percolation (the case of $g = 1$) is located at $h = 0$ and has different critical exponents: $\alpha = -1$, $\beta = 1$, $\tilde{\gamma} = 1$, and $\delta = 2$ [1]. Therefore in the MF approximation, scalar and rigidity percolation are in different universality classes. In two dimensions, a similar conclusion is reached by numerical means [48,15] where both transitions are continuous in zero field.

It has been recently argued [62] that for certain spin systems, the existence of a discontinuous transition in the MF approximation is enough to ensure that, in finite space dimensions d, there is a discontinuous transition if d is large enough. If this result holds for RP, one would expect to see a discontinuous transition in zero field for d (and perhaps g) large enough. Up to now, extensive numerical simulations for RP have been performed only in two dimensions [15,48,52,63]. Except for pathological cases where the transition happens at zero dilution [15], it seems that the transition is continuous for all g in two dimensions. Large scale simulations are still needed to clarify this issue.

On Cayley trees with a boundary, g-RP is closely related to $(g+1)$-BP [36,20]. Both percolate at the same density p of present bonds, although with different spanning cluster densities on trees. On random graphs, which have the local structure of a tree but no boundaries, the k-core transition stays unchanged, but the RP transition is delayed to larger p values [20]. We have explicitly shown that, at the critical concentration γ_c where the rigid cluster first appears, it has an exact balance of degrees of freedom, i.e., it has exactly g bonds per site. This condition is satisfied both on the whole rigid cluster and on its $(g+1)$-rigid core. However globally the system has lesser bonds than needed to attain this balance, meaning that the rigid cluster is selectively made out of sites with more bonds than average.

In previous work on Bethe lattices in zero field [21] it was suggested that the number n_F of uncanceled degrees of freedom is a free energy for the RP problem. For SP $(g = 1)$ this result holds exactly, being one of the outcomes of the Fortuin-Kasteleyn random-cluster model [46,47], which contains SP and the Potts model as particular cases. For RP, however, this identification only has the status of a plausible ansatz. This ansatz was used recently to predict some thermodynamic properties of chalcogenide glasses [64], for which the RP transition has been shown to be relevant [65,66,11,14].

Some of the reasons to believe that this identification might be correct in general are (a) the fact that n_F must be continuous at the discontinuous RP transition and (b) the fact that for $g = 1$, n_F is the number of connected clusters per site, so the Fortuin-Kasteleyn result for SP is recovered exactly. In this work we have shown that, in the presence of a properly defined field h, the order parameter R can be obtained as a derivative of the free energy with respect to h, thus adding further support to the belief that this identification is correct.

Under the assumption that $\ln \mathcal{Z} = N n_F$, the entropy per site s can be derived. The resulting expression (41) was found to depend on the order parameter R alone. This property is shared by other MF systems, for example, the Ising ferromagnet.

On the pedagogical side, we have attempted to situate the discussion in terms of the parallel between the RP transition in a field and a condensation transition. Since only topological (connectivity) properties are important for RP, it can be said that, in a sense, RP in a field is a sort of ''geometric condensation transition.'' Figure 7 illustrates the similarities between both transitions. However a closer match is possible. Comparing Eq. (7) to Eq. (9) one notes that, in a MF condensation transition, $\lambda \propto \rho^2 = 1/v^2$ plays the role of the order parameter R in RP. From this point of view, the analog of volume in RP would be $1/R^{1/2}$. The analog of a P-V plot is shown in Fig. 8. Note however that, within this definition of volume, $P(V)$ does *not* satisfy Maxwell's rule of equal areas, $\int_A^B V(P)dP = 0$.

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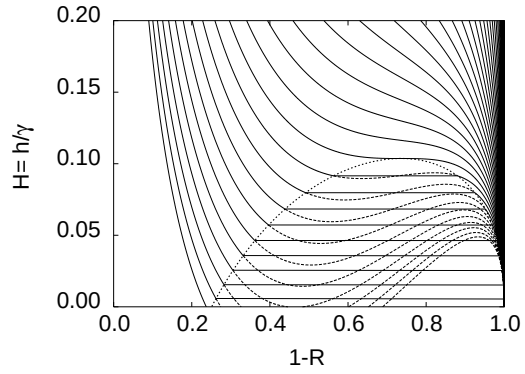

FIG. 7. This plot of the field H vs the order parameter R for rigidity percolation ($g=2$ in this example) is the equivalent of a P-V diagram for fluids. Shown are the isotherms (solid lines) along which the coordination number γ is constant. The thick dashed line delimits the ''coexistence'' region, and was obtained by solving Eq. (39) numerically. The areas delimited by the thin dashed lines above and below the (solid) horizontal coexistence line are equal.

APPENDIX A: ALTERNATIVE DEFINITIONS OF FIELD

Two different ways to introduce a field have been used in the percolation literature. We will call them the ''linear'' [29,32,33,35] and the ''Poissonian'' [30,31,34] fields. The linear definition states that each site may be either connected or not to infinity (or the background) by a ghost bond, with probabilities $\tilde{h}$ or $1-\tilde{h}$, respectively. In the Poissonian definition, each site can be connected to the background by $n=0,1,2,\ldots,\infty$ ghost bonds, and the probability to have n ghost bonds is $P_n(h)=e^{-h}h^n/n!$. In SP each site has one degree of freedom, so all that matters is whether the site is free from ghost bonds or not. In this case, both definitions are formally equivalent and related through a change of variables $\tilde{h}=1-e^{-h}$. However, note that it is the Poissonian definition which allows the most natural connection with thermodynamics, in the sense that it plays the exact role of a magnetic field in spin systems, when the number of clusters free from ghost bonds is identified as the free energy [34].

In the case of RP ($g>1$), the Poissonian definition stays unchanged, but the linear definition has to be modified slightly. A single ghost bond is not enough to rigidize a site, but one may generalize the definition by saying that a site is either connected or not to the background by g ghost bonds, respectively, with probabilities $\tilde{h}$ and $1-\tilde{h}$. Within the linear definition, ghost constraints are binary variables, that is, they may be in one of two states. In the Poissonian definition on the other hand, each site can be in one of $g+1$ states (there is no difference whatsoever between having g and more than g ghost constraints, so all these can be regarded as a single state). In this work we have chosen to use the Poissonian definition, mainly because it considerably simplifies calculations.

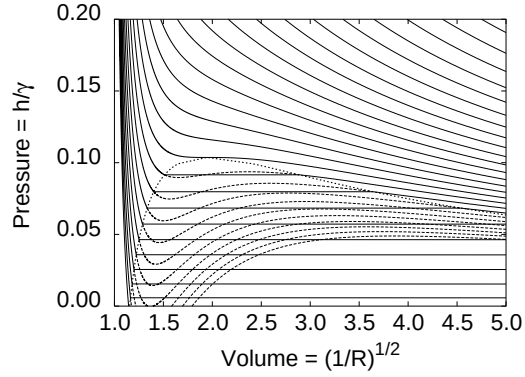

FIG. 8. Illustration of the similarities between RP and a condensation transition. In this plot, $H=h/\gamma$ is taken to be the analog of pressure and $1/R^{1/2}$ the analog of volume. The data are the same as shown in Fig. 7.

APPENDIX B: DERIVATIVES OF THE NUMBER OF REDUNDANT BONDS

Let $\mathcal{F}(\mathcal{B})$ be an arbitrary function of the set of present edges $\mathcal{B}$ of a graph $\mathcal{G}$, and let $P(\mathcal{B})$ be the probability to have edge set $\mathcal{B}$. The average of $\mathcal{F}$ is defined as

$$\langle \mathcal{F} \rangle = \sum_{\{\mathcal{B}\}} P(\mathcal{B}) \mathcal{F}(\mathcal{B}), \tag{B1}$$

where $\Sigma_{\{\mathcal{B}\}}$ is a sum over all configurations of present edges.

If each edge is independently present with probability p and absent with probability $1-p$, then clearly

$$P_p(\mathcal{B}) = p^{|\mathcal{B}|}(1-p)^{|\mathcal{B}_{max}|-|\mathcal{B}|}, \tag{B2}$$

where $\mathcal{B}_{max}$ is the edge set of maximum possible cardinality.

We now write $p=p_1p_2$ with $0 \leqslant p_1, p_2 \leqslant 1$. This can be realized by assuming that two types of bonds are independently present with probabilities p_1 and p_2 respectively, and that in order for an edge to be ''active,'' both types of bonds must be present on that edge. $\mathcal{F}$ is now a function of active bonds only, which are present with probability p_1p_2, so that one can write

$$\begin{aligned} \langle \mathcal{F} \rangle_{p_1p_2} &= \sum_{\{\mathcal{B}\}} P_{p_1}(\mathcal{B}) \sum_{\{\mathcal{B}'\} \subseteq \mathcal{B}} P_{p_2}(\mathcal{B}') \mathcal{F}(\mathcal{B}') \\ &= \sum_{\{\mathcal{B}\}} P_{p_1}(\mathcal{B}) S(p_2, \mathcal{B}). \end{aligned} \tag{B3}$$

Note that $P_{p_2}(\mathcal{B}')$ has a factor p_2 for each present bond in configuration $\mathcal{B}'$, and a factor $1-p_2=q_2$ for each bond in $\mathcal{B}$ which is absent in $\mathcal{B}'$. Factors associated with bonds not in $\mathcal{B}$ can be summed out to give one and therefore need not be considered. Thus if $|\mathcal{B}|$ is the number of present bonds in configuration $\mathcal{B}$ and $|\mathcal{B}'|$ the number in $\mathcal{B}'$, $P_{p_2}(\mathcal{B}') = p_2^{|\mathcal{B}'|} q_2^{|\mathcal{B}|-|\mathcal{B}'|}$.

Now let $p_2 \to 1$. In this case we can expand $S(p_2, \mathcal{B})$ in powers of q_2. The zeroth-order contribution comes from $\mathcal{B}' \equiv \mathcal{B}$, the first-order contribution from the $|\mathcal{B}|$ configurations $\mathcal{B}'$ which have exactly one bond b less than $\mathcal{B}$, etc.:

$$S(p_2,\mathcal{B}) = p_2^{|\mathcal{B}|}\mathcal{F}(\mathcal{B}) + q_2 p_2^{|\mathcal{B}|-1}\sum_{b\in\mathcal{B}}\mathcal{F}(\mathcal{B}-b) + q_2^2 p_2^{|\mathcal{B}|-2}\sum_{b,b'\in\mathcal{B}}\mathcal{F}(\mathcal{B}-b-b') + \cdots, \tag{B4}$$

where we have written $\mathcal{B}-b$ to denote the result of deleting bond b from $\mathcal{B}$. To first order in q_2,

$$S(p_2,\mathcal{B}) = \mathcal{F}(\mathcal{B}) + q_2\sum_{b\in\mathcal{B}}[\mathcal{F}(\mathcal{B}-b)-\mathcal{F}(\mathcal{B})], \tag{B5}$$

therefore

$$\langle\mathcal{F}\rangle_{p_1p_2} = \langle\mathcal{F}\rangle_{p_1} + q_2\left\langle\sum_{b\in\mathcal{B}}[\mathcal{F}(\mathcal{B}-b)-\mathcal{F}(\mathcal{B})]\right\rangle_{p_1}. \tag{B6}$$

Taking derivatives with respect to p_2 and letting $p_2=1$ one finally finds that, for arbitrary $\mathcal{F}$,

$$p\frac{\partial\langle\mathcal{F}\rangle}{\partial p} = \left\langle\sum_{b\in\mathcal{B}}[\mathcal{F}(\mathcal{B})-\mathcal{F}(\mathcal{B}-b)]\right\rangle. \tag{B7}$$

Equation (B7) generalizes previous results of Coniglio in Ref. [67], and is the tool we use in this section to derive some important relations.

Consider now the case in which $\mathcal{F}=B_r(\mathcal{B})$, the total number of redundant constraints. In order to calculate the derivative of B_r with respect to p, we note that when removing a bond b from a configuration $\mathcal{B}$ of present bonds, the total number of redundant constraints will be reduced by one if and only if b is overconstrained. Otherwise if b is not overconstrained, B_r remains unchanged by the removal of b. Equation (B7) then implies that

$$p\frac{\partial\langle B_r\rangle}{\partial p} = \gamma\frac{\partial\langle B_r\rangle}{\partial\gamma} = \langle B_{ov}\rangle, \tag{B8}$$

where $\langle B_{ov}\rangle$ is the average number of overconstrained bonds in $\mathcal{B}$.

A calculation of the field derivative of $\langle B_r\rangle$ requires some additional considerations. Equation (B7) was derived under the assumption that each edge is independently present with some fixed probability, while our definition of ghost field in Sec. II C implies that each site can have *any* number n of ghost constraints with Poissonian probability $P_n(h) = e^{-h}h^n/n!$. We can represent this Poissonian process in a way that enables us to use Eq. (B7) as follows. We assume that each site has $M \gg h$ *slots* connecting it to the background, and that each slot is occupied by *one* ghost edge with probability $p_h = h/M$. For large M, the probability to have n ghost edges on a site is $P_n(h)$, so one has a realization of the Poissonian distribution. This equivalent system has a total of NM *ghost slots*, each occupied by a ghost bond with a small probability p_h, so we can now use Eq. (B7).

Eliminating a present ghost edge reduces the number of redundant constraints by one if both the following conditions are satisfied: (a) the remaining constraints that this site has are enough to rigidize (i.e., $k_\gamma + k_h \geqslant g$) and (b) the remaining ghost constraints that this site has are not enough to rigidize on their own (i.e., $k_h < g$). The joint probability for these two conditions to be satisfied is

$$\begin{aligned}\sum_{j=0}^{g-1}P_j(h)\sum_{k=g-j}^{\infty}P_k(x) &= \sum_{j=0}^{g-1}P_j(h) - \sum_{j=0}^{g-1}P_j(h)\sum_{k=0}^{g-j-1}P_k(x)\\ &= R(\gamma,h) - R(0,h) = R - R_h,\end{aligned} \tag{B9}$$

where we have defined $R_h = G_g(h)$ as the density of rigid sites in the presence of the field h for a graph with no edges ($\gamma=0$). The total number of ghost slots is NM and thus the total number of present ghost bonds which upon removal produce a change in the number of redundant constraints is on average

$$NM\frac{h}{M}(R-R_h). \tag{B10}$$

Equation (B7), with $p_h = h/M$, now implies that

$$\frac{\partial\langle B_r\rangle}{\partial h} = N(R-R_h). \tag{B11}$$

Using $\langle B_r\rangle_\gamma = \gamma_r N/2$ one then has

$$\frac{\partial\gamma_r}{\partial h} = 2(R-R_h). \tag{B12}$$

Checking derivatives of γ_r

It is a trivial exercise to show that γ_r, as given by Eq. (28) has the right derivatives, i.e., satisfies

$$\frac{\partial\gamma_r}{\partial\gamma} = R^2 \tag{B13}$$

in agreement with Eq. (25), and

$$\frac{\partial\gamma_r}{\partial h} = 2\{R-R_h\}, \tag{B14}$$

in agreement with Eq. (B12). Now since $g_{\text{eff}} = g[1-G_{g+1}(h)] - h[1-G_g(h)]$ one has that $\partial g_{\text{eff}}/\partial h = G_g(h) - 1 = R_h - 1$. Therefore the derivatives of n_F [Eq. (22)] are

$$\frac{\partial n_F}{\partial\gamma} = \frac{R^2-1}{2},$$

$$\frac{\partial n_F}{\partial h} = R-1. \tag{B15}$$

APPENDIX C: DERIVATION OF CRITICAL INDICES

1. Scalar percolation ($g=1$)

Expanding $R=G_1(y)$ around $y^*=\gamma^*R^*+h^*$, and taking into account the condition $G_1'(x^*)=1/\gamma^*=1$ (Sec. III B) one has $R\approx\Delta y-1/2(\Delta y)^2$, with $\Delta y=y-y^*$. The critical point is at $y^*=0$ so one can write, to second order in $\Delta y=y=\gamma R+h$,

$$R=\gamma R+h-\frac{(\gamma R+h)^2}{2}. \tag{C1}$$

For $h=0$ this gives $R\approx 2(\gamma-1)$ so $\beta=1$.

On the critical isotherm $\gamma=1$ one has $R\approx R+h-1/2(R+h)^2$. For small h this reads $2h\approx(R+h)^2\approx R^2$, or $R\approx h^{1/2}$, from which $\delta=2$ results.

The susceptibility $\chi=\partial R/\partial h|_{h^*}$ diverges on approach to the critical point as $\chi\propto(\Delta\gamma)^{-\tilde{\gamma}}$. Deriving Eq. (7) one gets

$$\frac{\partial R}{\partial h}=\frac{P_{g-1}}{1-\gamma P_{g-1}}, \tag{C2}$$

and recalling that $\gamma^*=[P_{g-1}(y^*)]^{-1}$ we find that $\chi\propto(\Delta\gamma)^{-1}$, or $\tilde{\gamma}=1$.

2. Rigidity percolation ($g>1$)

In this case the critical point is an inflexion point of G_g, thus we need to expand to third order (this is in fact the reason why the universality class is not the same as for SP). In doing this, Eqs. (12a) and (12b) imply that, for $y\approx y^*$,

$$R\approx R^*+\frac{\Delta y}{\gamma^*}+O((\Delta y)^3), \tag{C3}$$

with $\Delta y=y-y^*$ and $y^*=\gamma^*R^*+h^*$. The coefficient of the $O((\Delta y)^3)$ term is nonzero. Furthermore $\Delta y=\Delta h+\gamma^*\Delta R+R\Delta\gamma$, and Eq. (C3) can be rewritten as

$$\gamma^*R=\gamma^*R^*+x-x^*+\Delta h+O((\Delta y)^3)\Rightarrow R\Delta\gamma+\Delta h=O((\Delta y)^3). \tag{C4}$$

On the critical isotherm ($\Delta\gamma=0$) we thus have that $\Delta h\sim(\gamma^*\Delta R+\Delta h)^3\Rightarrow\Delta R\approx(\Delta h)^{1/3}$, so $\delta=3$.

The exponent β is defined by the jump J_R that the order parameter suffers on the coexistence line. J_R behaves as ϵ^β for $\epsilon\to 0$, where $\epsilon>0$ parametrizes the coexistence line and $\epsilon=0$ on the critical point. Equivalently [50], β may be defined by the shape of the coexistence region in a P-V diagram, or its equivalent (see the H-R diagram of Fig. 7). The coexistence region widens as $|\Delta T|^\beta$ below T^*. Figure 7 shows a parabolic maximum, suggesting $\beta=1/2$. Nevertheless we derive this result explicitly in the following.

Equation (C3) [which approximates Eq. (7) in the neighborhood of the critical point] has three solutions $R_A<R_0<R_B$ for all $\{h,\gamma\}$ on the coexistence line. The discontinuity we want to calculate is $J_R=R_B-R_A$. The Claussius-Clapeyron relation (47) implies that, close to the critical point, the coexistence line is defined by $\Delta h=-R^*\Delta\gamma$. Plugging this result back into Eq. (C3) and eliminating the solution $R=R^*$ one finds that the other two solutions behave as $\Delta R\propto(\Delta\gamma)^{1/2}$, or $\beta=1/2$.

The derivation of $\tilde{\gamma}=1$ in the preceding section holds for RP as well, thus $\tilde{\gamma}=1$ also here. Note that Rushbrooke's $(\alpha+2\beta+\tilde{\gamma}=2)$ and Griffith's $(\alpha+\beta(\delta+1))=2)$ relations are satisfied with $\alpha=0$, and this is consistent with the fact that the constant-R specific heat (44) is zero.

APPENDIX D: BETHE LATTICES

A graph where each site has exactly z randomly chosen neighbors is topologically equivalent to a Bethe lattice, except for finite-size corrections. When this system is randomly diluted, one obtains what is called ''random bond model'' in Ref. [21], in which each of the z bonds on a site is present with probability p. In zero field, the density R of rigid sites is defined by [20]

$$R=\sum_{j=g}^{z}Q_j^{(z)}(x), \tag{D1}$$

where $x=pT$, $Q_j^{(z)}(x)=\binom{z}{j}x^j(1-x)^{z-j}$, and T satisfies the recursive equation

$$T=\sum_{j=g}^{z-1}Q_j^{(z-1)}(x). \tag{D2}$$

This last equation admits multiple solutions, so the continuity of the density of redundant bonds must be invoked in order to determine the transition point, as first discussed in Ref. [21]. The density of redundant bonds $r(p)$ is [21]

$$r(p)=\int_0^p T^2dp=1-\frac{2g}{z}-\int_p^1 T^2dp. \tag{D3}$$

Using the same procedure that was used for Erdös-Renyi graphs in the rest of this work, one can exactly integrate this last condition and conclude that, for zero field, the transition point p_c is determined by

$$\sum_{j=g+1}^{z}(j-2g)Q_j^{(z)}(x_c)=0. \tag{D4}$$

The transition condition (D4) gives rise to a polynomial equation which is easily solved for small values of g and z. For $g=2$ and $z=5$ the corresponding quadratic equation results in $p_c=0.834\,842\,34$. For $g=2$ and $z=6$ a cubic equation is obtained, and its solution is $p_c=0.656\,511\,134$. This last value is consistent with but more precise than $p_c=0.656$ as obtained in previous work [21] using numerically exact matching algorithms for RP [51,68].

Because Eq. (D2) has a different structure from Eq. (2), a Poissonian field as defined in Sec. II C does not seem adequate for this system. In practice, one finds that the equations for R in the presence of a Poissonian field cannot be put in a simple closed form, as is the case of Eq. (7) for Erdös-Renyi graphs. One may be tempted to try a ''binomial'' field

definition, in which each site has n ghost bonds to the background with probability $Q_n^{(z-1)}(h)$, however the resulting expressions do not simplify either. The simplest choice appears to be the linear field discussed in Appendix A, in which each site is rigidly attached to the background by g constraints with probability h, and free with probability $1-h$. In the presence of a linear field one has

$$T = h + (1-h)\sum_{j=g}^{z-1} Q_j^{(z-1)}(x) \tag{D5}$$

and

$$R = h + (1-h)\sum_{j=g}^{z} Q_j^{(z)}(x), \tag{D6}$$

while the density of redundant bonds reads

$$r(p,h) = \int_0^p T^2 dp = 1 - \frac{2g}{z}(1-h) - \int_p^1 T^2 dp, \tag{D7}$$

which can again be integrated exactly. The overall results are qualitatively the same as for Erdös-Renyi graphs, although the resulting analytic expressions are by far less elegant.

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